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Multifractality of Nonlinear Iterative Processes

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The Havlin–Bunde multifractal hypothesis [*Physica D* **38**:184 (1989)] is expanded (in the form of the dimension-invariance approach) to nonlinear iterative (recursion) processes such as dielectric breakdown, phase transitions from periodic attractors to chaos, and cascades in turbulence. Comparison with model and laboratory data of different authors shows that for strong nonlinearity the dimension invariance is broken.

KEY WORDS: Scaling; dimension invariance; dielectric breakdown; chaos.

In ref. 1 a generalization of multifractality is suggested in the form of a pseudo-scaling hypothesis. In the ref. 1 this hypothesis is rigorously proved for linear fractals and it is strongly supported for percolation systems by numerical simulations. In ref. 2, I showed that this hypothesis could be related to the mapping $m \to m^{\lambda}$, where m is the measure under consideration. In this note I develop a dimension-invariance approach based on this mapping and show that the pseudo-scaling hypothesis is applicable also to nonlinear iterative processes such as dielectric breakdown, phase transitions from periodic attractors to chaos in the Feigenbaum scenario, and critical cascades in turbulence. Moreover, comparison with model and laboratory data of different authors shows that for a strong nonlinearity the dimension invariance is broken. The main idea is based on the statement that there is not (in some sense) a *fixed* dimensionality in the multifractal processes, unlike the case for monofractal ones. Therefore some analogy with the scale-invariance approach could be possible (recall that the scale-invariance approach is based on the absence, in some sense, of

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a fixed scale in the system). Thus the multiscaling which appears in nonlinear iterative (recursion) processes[3] could simultaneously lead to scale invariance and to dimension invariance (multifractality).

1. Let us start with some standard definitions. Suppose that the total volume of a sample consist of a d-dimensional cube of size L. We divide this volume into N boxes of linear size $r[N \sim (L/r)^d]$. We label each box by the index i and construct for each box the measure function of a field $\mu(\mathbf{x}, t)$

$$m_i(r) = \int_{v_i} \mu(\mathbf{x})\, dv \tag{1}$$

where v_i is volume of the ith box. Then the generalized dimension D_q can be introduced by the following scaling relationship (see, for instance, ref. 3 and references therein)

$$Z_p = \sum_{i=1}^{N} [m_i(r)]^p \sim r^{(p-1)D_p} \tag{2}$$

The standard averaging is

$$\langle m(r)^p \rangle = \frac{\sum_{i=1}^{N} [m_i(r)]^p}{N} \sim r^{(p-1)D_p + d} \tag{3}$$

Such a description is generally useful for scale-invariant systems. One can use the scale-invariance approach if there is not a *fixed* scale in the system under consideration. This means the invariance of some relationships with scale (measure) stretching, such as

$$m \to \lambda m \tag{4}$$

This invariance leads to the power form of laws. On the other hand, in multifractal systems there is not a *fixed* dimensionality, and then one could expect that dimensionless relationships are invariant with power stretching of the measure

$$m \to m^{\lambda} \tag{5}$$

To use the dimension-invariance approach let us consider dimensionless moments

$$F_{np} = \frac{\langle m^p \rangle}{\langle m^n \rangle^{p/n}} \tag{6}$$

If there exists a pseudo-scaling relationship between the dimensionless moments[2]

$$F_{np} \sim (F_{nq})^{\rho(n,\,p,\,q)} \tag{7}$$

then it is easy to show that the dimension invariance leads to the relationship

$$\rho(\lambda n, \lambda p, \lambda q) = \rho(n, p, q) \tag{8}$$

On the other hand, this condition leads to a reduction of the number of independent variables from three to two, which can be written in the form

$$\rho(n, p, q) = f\left(\frac{p}{n}, \frac{q}{n}\right) \tag{9}$$

where $f(x, y)$ is some function.

It is clear that the pseudo-scaling (7) always exists for systems with ordinary scaling (3). If we denote

$$D_q(q-1) + d = q\beta(q) \tag{10}$$

then it follows from (3), (7), (9), and (10) that

$$\frac{\beta(p) - \beta(n)}{\beta(q) - \beta(n)} = \frac{q/n}{p/n} f\left(\frac{p}{n}, \frac{q}{n}\right) \tag{11}$$

This functional equation for $\beta(p)$ has two solutions:

$$\beta(p) = a + bp^{\gamma} \tag{12}$$

and

$$\beta(p) = a + b \ln p \tag{13}$$

where a, b, and γ are some constants.

The first solution, (12), is related to the original Havlin–Bunde hypothesis.[1] The second solution, (13), is an additional dimension-invariant form of the scaling exponents and we shall study just this form in more detail below.[2,4]

2. It should be noted that appearance of the topological dimension d in (10) is in contradiction with the main idea of the dimension-invariance approach. Generally speaking, this approach should not be applicable even in some vicinity of the capacity dimension D_0. To solve this "paradox," let us introduce an effective averaging

$$\langle m(r)^p \rangle_{ef} = \frac{\sum_{i=1}^{N} [m_i(r)]^p}{N_{ef}} \sim r^{(p-1) D_p + D_{ef}} \tag{14}$$

instead of the standard averaging (3), where $N_{ef} = (L/r)^{D_{ef}}$ and D_{ef} is some effective dimension. This effective (virtual) dimension should be obtained in a self-consistent way from the dimension-invariance approach itself, and it has no simple geometrical meaning. Then, using the effective averaging (14) instead of (3), we obtain for the dimension-invariance approach

$$D_q(q-1) + D_{ef} = q\beta(q) \tag{15}$$

with $\beta(q)$ given by (12) or (13); and $q = 0$ is a singular point of this approach.

Analogously, the dimension-invariance approach should not be applicable for large values of $|q|$, where other fixed dimensions, D_∞ and $D_{-\infty}$, could be relevant. The situation is quite analogous to the situation with scale invariance. The scale-invariance approach is applicable in some interval of scales $L > r > l$, where L and l are some *fixed* outer and inner scales of the system.

If points $|q| = 1$ belong to the interval of the region of applicability of the dimension-invariance approach, we obtain from (15) and (13)

$$D_q = D_{-1} + b \frac{q \ln |q|}{(q-1)}$$

Since point $q = 0$ is a singular point of the approach,

$$D_q = D_{-1} + b_+ \frac{q \ln |q|}{(q-1)} \qquad (b_+ < 0 \text{ for } q > 0) \tag{16a}$$

$$D_q = D_{-1} + b_- \frac{q \ln |q|}{(q-1)} \qquad (b_- > 0 \text{ for } q < 0) \tag{16b}$$

This representation could be applicable even for the situation where the point $q = 1$ does not belong to the region of applicability of the dimension-invariance approach due to elimination of D_q from relationship (15) at $q = 1$.

3. Let us demonstrate the applicability of this approach. In ref. 5 deterministic fractal models of dielectric breakdown are presented and the recursion relations of the electric field on the growth bond are obtained.

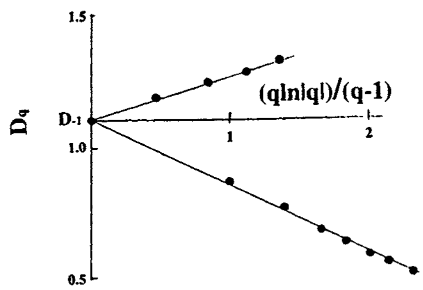

Fig. 1. A Spectrum of generalized dimension D_q (adapted from ref. 5) for the dielectric breakdown model with $z = 2$. The straight lines are drawn for comparison with the dimension-invariant representation (16).

The growth probability p_i at the growing perimeter bond i is given by $p_i \sim (E_i)^z$, where E_i is the local electric field at the growth bond. The generalized dimensions D_q are calculated in ref. 5 (see also ref. 6) to describe the growth probability, by using the recursion relations. Figure 1 (adapted from ref. 5) shows the set of D_q against $q \ln |q|/(q-1)$ for $z = 2$. Straight lines are drawn in Fig. 1 for comparison with the dimension-invariance relationship (16) (upper set of symbols corresponds to $q < 0$ and lower set corresponds to $q > 0$). One can see good agreement between these data and the dimension-invariance approach. For $z = 4$, however, there is disagreement between the dimension-invariance approach and the D_q calculated in ref. 5. This could be an indication that for strong nonlinearity the dimension invariance is broken (see also additional evidence of this phenomenon below).

Phase transitions from periodic attractors to chaos in the Feigenbaum scenario also exhibit dimension invariance. Calculated in ref. 7 were the generalized dimensions D_q for critical strange sets which refer to the Feigenbaum-type attractors formed at critical points of transitions to chaos in 1D iterative system.[8] Fig. 2 (adapted from ref. 7 shows these generalized dimensions calculated for the map $f(x) = 1 - a\,|x|^z$ for $z = 2$. Again one can see good agreement between the data and the dimension-invariance representation (16) (note that D_q in this case is normalized on D_0; ref. 7). Increasing nonlinearity leads to violation of the dimension invariance for this process as well. Indeed, calculation performed in ref. 7 for $z = 3$ gives a set of generalized dimensions which does not exhibit the dimension-invariance properties (cf. previous case).

Another interesting example of a nonlinear iterative process with the nonlinearity index $z = 2$ is given by critical formation of cascading turbulence (see, for instance, refs. 3, 9, 10, and references therein).

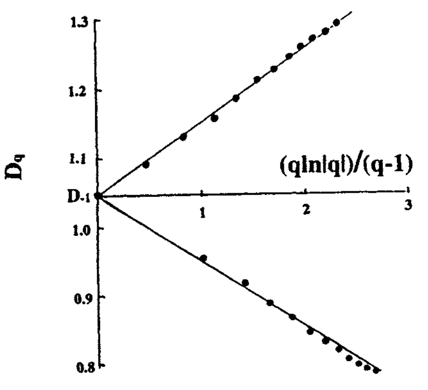

Fig. 2. A spectrum of generalized dimension D_q (normalized on D_0, adapted from ref. 7) for the critical strange sets which hereafter refer to Feigenbaum-type attractors formed on the critical points of the transition to chaos ($z = 2$). The straight lines are drawn for comparison with the dimension-invariant representation (16).

Figure 3 shows (averaged) experimental data obtained in atmospheric boundary layers, in a laboratory turbulent boundary layer, in the turbulent wake of a circular cylinder, and in a turbulent flow behind a square grid of round bars (the data taken from ref. 3 and based on experiments[11, 12]). We do not show the data belonging to a vicinity of the point $q = 0$, where the dimension-invariance approach becomes nonapplicable. Again one can see good agreement with the dimension-invariance representation (16)

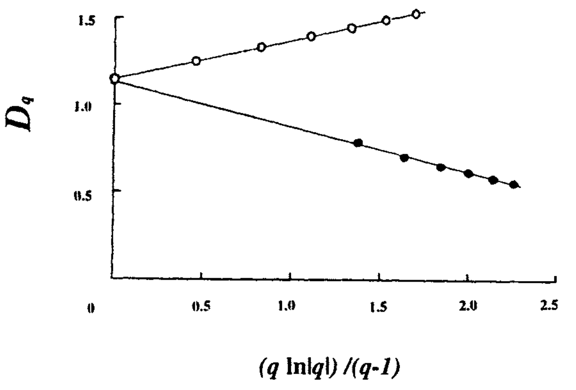

Fig. 3. A spectrum of (averaged) generalized dimension D_q of turbulent energy dissipation obtained in different turbulent flows (data taken from refs. 3, 11, and 12). The straight lines are drawn for comparison with the dimension-invariant representation (16) [the point with $(q \ln |q|)/(q-1) = 0$ corresponds to D_{-1}].

(cf. Figs. 1 and 2). In this case (unlike previous ones) the index of nonlinearity, $z=2$, cannot be varied. However, there are many analogous stochastic processes where the case $z>2$ could be studied and it seems to be a rather intriguing problem whether this value of the nonlinear index is a critical one for the applicability of the dimension-invariance approach.

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